# Second-price auction with two bidders

Consider a second-price auction with three bidders. Their valuations are, respectively,

$$8,\ 5,\ 3$$

Assume that, in the event of a tie, the object is assigned to the bidder with the highest valuation.

1. Determine the payoffs of the bidders as a function of the bid profile
2. Indicate whether the bid profile $(b_1, b_2, b_3) = (3, 3, 25)$ constitutes a Nash equilibrium. Is it reasonable?
3. Find, if they exist, the weakly dominant bids
4. Taking into account the result from part c), is there any Nash equilibrium that may be considered a focal point?

## Solution

1. Let $(b_1, b_2, b_3)$ be the bid profile. In a second-price auction, the bidder who wins pays the highest rejected bid

   Since ties are broken in favor of the bidder with the highest valuation, the priority order in ties is

$$1 \succ 2 \succ 3$$

   because

$$8 > 5 > 3$$

   Thus, bidder 1 wins any tie involving bidder 1, bidder 2 wins a tie with bidder 3 whenever bidder 1 is not tied, and bidder 3 wins only if she is the unique highest bidder

   The payoff of each bidder is equal to valuation minus payment if she wins, and equal to 0 if she loses

   **Bidder** 1

   Bidder 1 wins in the following cases:

$$b_1 > \max\{b_2, b_3\}$$

   or

$$b_1 = b_2 > b_3$$

   or

$$b_1 = b_3 > b_2$$

   or

$$b_1 = b_2 = b_3$$

   If bidder 1 wins uniquely, she pays the highest of the other two bids, namely $\max\{b_2, b_3\}$

   If bidder 1 wins after a tie at the highest bid, she pays that common highest bid, namely $b_1$

   Therefore,

$$u_1(b_1, b_2, b_3) = \begin{cases} 8 - \max\{b_2, b_3\} & \text{if } b_1 > \max\{b_2, b_3\} \\ 8 - b_1 & \text{if } b_1 = b_2 > b_3 \\ 8 - b_1 & \text{if } b_1 = b_3 > b_2 \\ 8 - b_1 & \text{if } b_1 = b_2 = b_3 \\ 0 & \text{otherwise} \end{cases}$$

   **Bidder** 2

   Bidder 2 wins only if she is the unique highest bidder, or if she ties with bidder 3 at the highest bid and bidder 1 is below them

   So bidder 2 wins when

$$b_2 > \max\{b_1, b_3\}$$

   or

$$b_2 = b_3 > b_1$$

If bidder 2 wins uniquely, she pays $\max\{b_1, b_3\}$
If bidder 2 wins after a tie with bidder 3, she pays the common highest bid $b_2$
Therefore,

$$u_2(b_1, b_2, b_3) = \begin{cases} 5 - \max\{b_1, b_3\} & \text{if } b_2 > \max\{b_1, b_3\} \\ 5 - b_2 & \text{if } b_2 = b_3 > b_1 \\ 0 & \text{otherwise} \end{cases}$$

**Bidder 3**
Because bidder 3 has the lowest valuation, she wins only if she is the unique highest bidder
Thus bidder 3 wins when

$$b_3 > \max\{b_1, b_2\}$$

and in that case she pays $\max\{b_1, b_2\}$
Therefore,

$$u_3(b_1, b_2, b_3) = \begin{cases} 3 - \max\{b_1, b_2\} & \text{if } b_3 > \max\{b_1, b_2\} \\ 0 & \text{otherwise} \end{cases}$$

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2. Consider the bid profile

$$(b_1, b_2, b_3) = (3, 3, 25)$$

Since bidder 3 submits the unique highest bid, bidder 3 wins the object
In a second-price auction, the winner pays the highest rejected bid, which here is

$$\max\{3, 3\} = 3$$

Therefore, the payoffs are

$$u_1 = 0$$

$$u_2 = 0$$

$$u_3 = 3 - 3 = 0$$

We now check whether any bidder has a profitable unilateral deviation

**Bidder** 1

If bidder 1 keeps her bid below 25, she still loses and gets payoff

$$0$$

If bidder 1 deviates to

$$b_1' = 25$$

then she ties with bidder 3, and since ties are broken in favor of the bidder with the highest valuation, bidder 1 wins because

$$8 > 3$$

But then she pays 25, so her payoff is

$$8 - 25 = -17$$

If bidder 1 deviates to some bid $b_1' > 25$, she also wins and pays 25, so her payoff is again

$$8 - 25 = -17$$

Thus bidder 1 has no profitable deviation

**Bidder** 2

If bidder 2 keeps her bid below 25, she loses and gets payoff

$$0$$

If bidder 2 deviates to

$$b_2' = 25$$

then she ties with bidder 3, and wins the tie because

$$5 > 3$$

But then she pays 25, so her payoff is

$$5 - 25 = -20$$

If bidder 2 deviates to some bid $b_2' > 25$, she wins and still pays 25, so her payoff is again

$$5 - 25 = -20$$

Thus bidder 2 has no profitable deviation

**Bidder 3**

Bidder 3 currently gets payoff

$$0$$

If bidder 3 deviates to any bid strictly above 3, for example $b_3' = 4$, she still wins and still pays 3, so her payoff remains

$$3 - 3 = 0$$

If bidder 3 deviates to $b_3' = 3$, then all three bidders tie at 3, and bidder 1 wins because she has the highest valuation, so bidder 3's payoff is

$$0$$

If bidder 3 deviates to any bid below 3, she loses and also gets

$$0$$

Thus bidder 3 has no profitable deviation

Therefore, no bidder can improve her payoff by deviating unilaterally, so

$$(3, 3, 25) \text{ is a Nash equilibrium}$$

However, this equilibrium is not very reasonable

The reason is that bidder 3, whose valuation is only

$$v_3 = 3$$

is bidding

$$25$$

which is far above her valuation

Although this does not hurt her at this exact bid profile, it is a very fragile and implausible strategy, because if the rivals changed their bids slightly, bidder 3 could end up winning at a price above her valuation and obtaining a negative payoff

So this equilibrium exists mathematically, but it is not compelling economically

$$\text{Therefore, } (3, 3, 25) \text{ is a Nash equilibrium, but it is not a reasonable one}$$

3. We claim that, for each bidder $i$, the weakly dominant bid is to bid her true valuation

$$b_1 = 8, \qquad b_2 = 5, \qquad b_3 = 3$$

To justify this rigorously, fix a bidder $i$, let $v_i$ be her valuation, and let

$$m_i = \max\{b_j : j \neq i\}$$

be the highest bid submitted by the other bidders

In a second-price auction, if bidder $i$ wins, she pays $m_i$. Therefore, the only relevant comparison is between $m_i$ and $v_i$

**Case 1:** $m_i < v_i$
If bidder $i$ bids truthfully, that is,

$$b_i = v_i$$

then she wins the object and pays $m_i$, obtaining payoff

$$v_i - m_i > 0$$

Now compare this with any alternative bid $b_i'$
If

$$b_i' > m_i$$

bidder $i$ still wins and still pays $m_i$, so the payoff remains

$$v_i - m_i$$

If

$$b_i' \leq m_i$$

then bidder $i$ either loses or, in case of a tie, may fail to win because of the tie-breaking rule. In either situation, her payoff is at most 0, which is weakly smaller than

$$v_i - m_i$$

Thus, when $m_i < v_i$, truthful bidding is at least as good as any other bid

**Case 2:** $m_i > v_i$
If bidder $i$ bids truthfully,

$$b_i = v_i$$

then she does not submit a bid above $m_i$, so she loses or at best does not win the object, obtaining payoff

$$0$$

Now compare this with any alternative bid $b_i'$
If

$$b_i' \leq m_i$$

then bidder $i$ still does not win, so the payoff remains

$$0$$

If

$$b_i' > m_i$$

then bidder $i$ wins and pays $m_i$, so her payoff becomes

$$v_i - m_i < 0$$

Hence, when $m_i > v_i$, truthful bidding is again at least as good as any alternative bid

**Case 3:** $m_i = v_i$
If bidder $i$ bids truthfully,

$$b_i = v_i$$

then she either ties at the highest bid or exactly matches the highest opposing bid. If she wins, she pays $m_i = v_i$, so the payoff is

$$v_i - m_i = 0$$

If she loses the tie, the payoff is also

$$0$$

Now consider any alternative bid $b_i'$
If

$$b_i' < v_i$$

then bidder $i$ loses, so the payoff is

$$0$$

If

$$b_i' = v_i$$

the payoff is again

$$0$$

If

$$b_i' > v_i$$

then bidder $i$ may win, but she still pays $m_i = v_i$, so the payoff is

$$v_i - v_i = 0$$

Thus, when $m_i = v_i$, truthful bidding is also at least as good as any other bid

Since these three cases exhaust all possibilities, bidding one's true valuation is a weakly dominant strategy for every bidder
Therefore, in this exercise, the weakly dominant bids are

$$b_1 = 8, \qquad b_2 = 5, \qquad b_3 = 3$$

The intuition is standard: in a second-price auction, bidding higher than one's valuation may force the bidder to win at a loss, while bidding lower than one's valuation may cause the bidder to lose an object that would have generated positive surplus. Truthful bidding avoids both problems

Hence, each bidder's weakly dominant bid is her true valuation

4. From part (3), each bidder has a weakly dominant bid, namely her true valuation

$$b_1 = 8 \qquad b_2 = 5 \qquad b_3 = 3$$

Therefore, the bid profile

$$(8, 5, 3)$$

is a Nash equilibrium

Indeed, bidder 1 wins because she submits the highest bid, and in a second-price auction she pays the highest rejected bid, which is

$$5$$

Hence, the payoffs are

$$u_1 = 8 - 5 = 3 \qquad u_2 = 0 \qquad u_3 = 0$$

This equilibrium is a natural *focal point*

The reason is that it is the equilibrium that arises when every bidder follows the weakly dominant strategy identified in part (3). In contrast with equilibria such as $(3, 3, 25)$, it does not rely on extreme or implausible bids. Each bidder simply bids her own valuation, which is the most transparent and strategically robust behavior in a second-price auction

Thus, among the many Nash equilibria that may exist, the most salient one is

$$(8, 5, 3)$$

and it may be considered the focal equilibrium

Therefore, the focal Nash equilibrium is the truthful-bidding profile $(8, 5, 3)$

Remember that a focal point is an equilibrium that players are especially likely to select because it is salient, natural, or particularly easy to recognize among many possible equilibria